THE AXIOMS OF THE WHITEHEAD RUSSELL CALCULUS

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Abstract. This module notates the original axioms of the Whitehead-Russell calculus, the so called 'primitive propositions'. These five primitive propositions could be deduced by our four axioms.

Contents

module specification

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This module has the following specification:

Name:	proptheo2
Version:	1.00.00
Rule version:	1.00.00
Orgin:	`http://www.meyling.com/principia/0_00_51/proptheo2_1.00.00_1.00.00.qedeq`

The following modules were used:

Name:	propaxiom
Version:	1.00.00
Rule version:	1.00.00
Orgin:	`propaxiom_1.00.00_1.00.00.qedeq`
pdf:	`propaxiom_1.00.00_1.00.00.pdf`

At first we show a little proposition to demonstrate the basic proof methods of propositional calculus:

Theorem 0.1.

$$(P \Rightarrow (Q \lor P))$$

Proof.

1	$((P \Rightarrow Q) \Rightarrow ((A \lor P) \Rightarrow (A \lor Q)))$	add axiom axiom4
2	$((B \Rightarrow Q) \Rightarrow ((A \lor B) \Rightarrow (A \lor Q)))$	replace P by B in 1
3	$((B \Rightarrow (Q \lor P)) \Rightarrow ((A \lor B) \Rightarrow (A \lor (Q \lor P))))$	replace Q by $(Q \lor P)$ in 2

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4	$(((P \lor Q) \Rightarrow (Q \lor P)) \Rightarrow ((A \lor (P \lor Q)) \Rightarrow (A \lor (Q \lor P))))$	replace B by $(P \lor Q)$ in 3
5	$(((P \lor Q) \Rightarrow (Q \lor P)) \Rightarrow ((\neg P \lor (P \lor Q)) \Rightarrow (\neg P \lor (Q \lor P))))$	replace A by $\neg P$ in 4
6	$(((P \lor Q) \Rightarrow (Q \lor P)) \Rightarrow ((P \Rightarrow (P \lor Q)) \Rightarrow (\neg P \lor (Q \lor P))))$	reverse abbreviation impl in 5 at occurence 1
7	$(((P \lor Q) \Rightarrow (Q \lor P)) \Rightarrow ((P \Rightarrow (P \lor Q)) \Rightarrow (P \Rightarrow (Q \lor P))))$	reverse abbreviation impl in 6 at occurence 1
8	$((P \lor Q) \Rightarrow (Q \lor P))$	add axiom axiom3
9	$((P \Rightarrow (P \lor Q)) \Rightarrow (P \Rightarrow (Q \lor P)))$	MP with 8, 7
10	$(P \Rightarrow (P \lor Q))$	add axiom axiom2
11	$(P \Rightarrow (Q \lor P))$	MP with 10, 9

□

This is the first primitive proposition, its equal to our first axiom:

Theorem 0.2.

$$((P \lor P) \Rightarrow P)$$

Proof.

1	$((P \lor P) \Rightarrow P)$	add axiom axiom1

□

Now comes the second primitive proposition. It looks simular to our second axiom, but we have to use our first proposition to prove it:

Theorem 0.3.

$$(Q \Rightarrow (P \lor Q))$$

Proof.

1	$(P \Rightarrow (Q \lor P))$	add sentence lem1
2	$(P \Rightarrow (A \lor P))$	replace Q by A in 1
3	$(Q \Rightarrow (A \lor Q))$	replace P by Q in 2
4	$(Q \Rightarrow (P \lor Q))$	replace A by P in 3

□

The third primitive proposition:

Theorem 0.4.

$$((P \lor Q) \Rightarrow (Q \lor P))$$

Proof.

1	$((P \lor Q) \Rightarrow (Q \lor P))$	add axiom axiom3

□

The fourth primitive proposition was proved with the other primitve propositions by P. Bernays. Here comes the sledgehammer:

Theorem 0.5.

$$((P \vee (Q \vee A)) \Rightarrow (Q \vee (P \vee A)))$$

Proof.

1	$((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$	add axiom axiom4
2	$((P_7 \Rightarrow Q) \Rightarrow ((A \vee P_7) \Rightarrow (A \vee Q)))$	replace P by P_7 in 1
3	$((P_7 \Rightarrow P_8) \Rightarrow ((A \vee P_7) \Rightarrow (A \vee P_8)))$	replace Q by P_8 in 2
4	$((P_7 \Rightarrow P_8) \Rightarrow ((P_9 \vee P_7) \Rightarrow (P_9 \vee P_8)))$	replace A by P_9 in 3
5	$(P \Rightarrow (Q \vee P))$	add sentence lem1
6	$((P \Rightarrow (Q \vee P)) \Rightarrow ((A \vee P) \Rightarrow (A \vee (Q \vee P))))$	replace Q by $(Q \vee P)$ in 1
7	$((A \vee P) \Rightarrow (A \vee (Q \vee P)))$	MP with 5, 6
8	$(((A \vee P) \Rightarrow P_8) \Rightarrow ((P_9 \vee (A \vee P)) \Rightarrow (P_9 \vee P_8)))$	replace P_7 by $(A \vee P)$ in 4
9	$(((A \vee P) \Rightarrow (A \vee (Q \vee P))) \Rightarrow ((P_9 \vee (A \vee P)) \Rightarrow (P_9 \vee (A \vee (Q \vee P)))))$	replace P_8 by $(A \vee (Q \vee P))$ in 8
10	$((P_9 \vee (A \vee P)) \Rightarrow (P_9 \vee (A \vee (Q \vee P))))$	MP with 7, 9
11	$((Q \vee (A \vee P)) \Rightarrow (Q \vee (A \vee (Q \vee P))))$	replace P_9 by Q in 10
12	$((P \vee Q) \Rightarrow (Q \vee P))$	add axiom axiom3
13	$((D \vee Q) \Rightarrow (Q \vee D))$	replace P by D in 12
14	$((D \vee (A \vee (Q \vee P))) \Rightarrow ((A \vee (Q \vee P)) \vee D))$	replace Q by $(A \vee (Q \vee P))$ in 13
15	$((Q \vee (A \vee (Q \vee P))) \Rightarrow ((A \vee (Q \vee P)) \vee Q))$	replace D by Q in 14
16	$(((Q \vee (A \vee (Q \vee P))) \Rightarrow P_8) \Rightarrow ((P_9 \vee (Q \vee (A \vee (Q \vee P)))) \Rightarrow (P_9 \vee P_8)))$	replace P_7 by $(Q \vee (A \vee (Q \vee P)))$ in 4
17	$(((Q \vee (A \vee (Q \vee P))) \Rightarrow ((A \vee (Q \vee P)) \vee Q)) \Rightarrow ((P_9 \vee (Q \vee (A \vee (Q \vee P)))) \Rightarrow (P_9 \vee ((A \vee (Q \vee P)) \vee Q))))$	replace P_8 by $((A \vee (Q \vee P)) \vee Q)$ in 16
18	$((P_9 \vee (Q \vee (A \vee (Q \vee P)))) \Rightarrow (P_9 \vee ((A \vee (Q \vee P)) \vee Q)))$	MP with 15, 17
19	$((\neg(Q \vee (A \vee P)) \vee (Q \vee (A \vee (Q \vee P)))) \Rightarrow (\neg(Q \vee (A \vee P)) \vee ((A \vee (Q \vee P)) \vee Q)))$	replace P_9 by $\neg(Q \vee (A \vee P))$ in 18
20	$(((Q \vee (A \vee P)) \Rightarrow (Q \vee (A \vee (Q \vee P)))) \Rightarrow (\neg(Q \vee (A \vee P)) \vee ((A \vee (Q \vee P)) \vee Q)))$	reverse abbreviation impl in 19 at occurence 1
21	$(((Q \vee (A \vee P)) \Rightarrow (Q \vee (A \vee (Q \vee P)))) \Rightarrow ((Q \vee (A \vee P)) \Rightarrow ((A \vee (Q \vee P)) \vee Q)))$	reverse abbreviation impl in 20 at occurence 1
22	$((Q \vee (A \vee P)) \Rightarrow ((A \vee (Q \vee P)) \vee Q))$	MP with 11, 21
23	$(P \Rightarrow (P \vee Q))$	add axiom axiom2
24	$(A \Rightarrow (A \vee Q))$	replace P by A in 23
25	$(A \Rightarrow (A \vee P))$	replace Q by P in 24
26	$(Q \Rightarrow (Q \vee P))$	replace A by Q in 25
27	$(P \Rightarrow (A \vee P))$	replace Q by A in 5
28	$((Q \vee P) \Rightarrow (A \vee (Q \vee P)))$	replace P by $(Q \vee P)$ in 27
29	$(((Q \vee P) \Rightarrow P_8) \Rightarrow ((P_9 \vee (Q \vee P)) \Rightarrow (P_9 \vee P_8)))$	replace P_7 by $(Q \vee P)$ in 4
30	$(((Q \vee P) \Rightarrow (A \vee (Q \vee P))) \Rightarrow ((P_9 \vee (Q \vee P)) \Rightarrow (P_9 \vee (A \vee (Q \vee P)))))$	replace P_8 by $(A \vee (Q \vee P))$ in 29
31	$((P_9 \vee (Q \vee P)) \Rightarrow (P_9 \vee (A \vee (Q \vee P))))$	MP with 28, 30
32	$((\neg Q \vee (Q \vee P)) \Rightarrow (\neg Q \vee (A \vee (Q \vee P))))$	replace P_9 by $\neg Q$ in 31
33	$((Q \Rightarrow (Q \vee P)) \Rightarrow (\neg Q \vee (A \vee (Q \vee P))))$	reverse abbreviation impl in 32 at occurence 1

34	$((Q \Rightarrow (Q \lor P)) \Rightarrow (Q \Rightarrow (A \lor (Q \lor P))))$	reverse abbreviation impl in 33 at occurence 1
35	$(Q \Rightarrow (A \lor (Q \lor P)))$	MP with 26, 34
36	$((Q \Rightarrow P_8) \Rightarrow ((P_9 \lor Q) \Rightarrow (P_9 \lor P_8)))$	replace P_7 by Q in 4
37	$((Q \Rightarrow (A \lor (Q \lor P))) \Rightarrow ((P_9 \lor Q) \Rightarrow (P_9 \lor (A \lor (Q \lor P)))))$	replace P_8 by $(A \lor (Q \lor P))$ in 36
38	$((P_9 \lor Q) \Rightarrow (P_9 \lor (A \lor (Q \lor P))))$	MP with 35, 37
39	$(((A \lor (Q \lor P)) \lor Q) \Rightarrow ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P))))$	replace P_9 by $(A \lor (Q \lor P))$ in 38
40	$((P \lor P) \Rightarrow P)$	add axiom axiom1
41	$(((A \lor (Q \lor P)) \lor (A \lor (Q \lor P))) \Rightarrow (A \lor (Q \lor P)))$	replace P by $(A \lor (Q \lor P))$ in 40
42	$((((A \lor (Q \lor P)) \lor (A \lor (Q \lor P))) \Rightarrow P_8) \Rightarrow ((P_9 \lor ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (P_9 \lor P_8)))$	replace P_7 by $((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))$ in 4
43	$((((A \lor (Q \lor P)) \lor (A \lor (Q \lor P))) \Rightarrow (A \lor (Q \lor P))) \Rightarrow ((P_9 \lor ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (P_9 \lor (A \lor (Q \lor P)))))$	replace P_8 by $(A \lor (Q \lor P))$ in 42
44	$((P_9 \lor ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (P_9 \lor (A \lor (Q \lor P))))$	MP with 41, 43
45	$((\neg((A \lor (Q \lor P)) \lor Q) \lor ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (\neg((A \lor (Q \lor P)) \lor Q) \lor (A \lor (Q \lor P))))$	replace P_9 by $\neg((A \lor (Q \lor P)) \lor Q)$ in 44
46	$((((A \lor (Q \lor P)) \lor Q) \Rightarrow ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (\neg((A \lor (Q \lor P)) \lor Q) \lor (A \lor (Q \lor P))))$	reverse abbreviation impl in 45 at occurence 1
47	$((((A \lor (Q \lor P)) \lor Q) \Rightarrow ((A \lor (Q \lor P)) \lor (A \lor (Q \lor P)))) \Rightarrow (((A \lor (Q \lor P)) \lor Q) \Rightarrow (A \lor (Q \lor P))))$	reverse abbreviation impl in 46 at occurence 1
48	$(((A \lor (Q \lor P)) \lor Q) \Rightarrow (A \lor (Q \lor P)))$	MP with 39, 47
49	$((((A \lor (Q \lor P)) \lor Q) \Rightarrow P_8) \Rightarrow ((P_9 \lor ((A \lor (Q \lor P)) \lor Q)) \Rightarrow (P_9 \lor P_8)))$	replace P_7 by $((A \lor (Q \lor P)) \lor Q)$ in 4
50	$((((A \lor (Q \lor P)) \lor Q) \Rightarrow (A \lor (Q \lor P))) \Rightarrow ((P_9 \lor ((A \lor (Q \lor P)) \lor Q)) \Rightarrow (P_9 \lor (A \lor (Q \lor P)))))$	replace P_8 by $(A \lor (Q \lor P))$ in 49
51	$((P_9 \lor ((A \lor (Q \lor P)) \lor Q)) \Rightarrow (P_9 \lor (A \lor (Q \lor P))))$	MP with 48, 50
52	$((\neg(Q \lor (A \lor P)) \lor ((A \lor (Q \lor P)) \lor Q)) \Rightarrow (\neg(Q \lor (A \lor P)) \lor (A \lor (Q \lor P))))$	replace P_9 by $\neg(Q \lor (A \lor P))$ in 51
53	$(((Q \lor (A \lor P)) \Rightarrow ((A \lor (Q \lor P)) \lor Q)) \Rightarrow (\neg(Q \lor (A \lor P)) \lor (A \lor (Q \lor P))))$	reverse abbreviation impl in 52 at occurence 1
54	$(((Q \lor (A \lor P)) \Rightarrow ((A \lor (Q \lor P)) \lor Q)) \Rightarrow ((Q \lor (A \lor P)) \Rightarrow (A \lor (Q \lor P))))$	reverse abbreviation impl in 53 at occurence 1
55	$((Q \lor (A \lor P)) \Rightarrow (A \lor (Q \lor P)))$	MP with 22, 54
56	$((P_7 \lor (A \lor P)) \Rightarrow (A \lor (P_7 \lor P)))$	replace Q by P_7 in 55
57	$((P_7 \lor (Q \lor P)) \Rightarrow (Q \lor (P_7 \lor P)))$	replace A by Q in 56
58	$((P_7 \lor (Q \lor A)) \Rightarrow (Q \lor (P_7 \lor A)))$	replace P by A in 57
59	$((P \lor (Q \lor A)) \Rightarrow (Q \lor (P \lor A)))$	replace P_7 by P in 58

□

The fifth primitive proposition is our fourth axiom:

Theorem 0.6.

$$((P \Rightarrow Q) \Rightarrow ((A \lor P) \Rightarrow (A \lor Q)))$$

Proof.

1 $((P \Rightarrow Q) \Rightarrow ((A \vee P) \Rightarrow (A \vee Q)))$ add axiom axiom4

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